On the Design of Reduced-order H_2 Controllers with Coefficient Constraint

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Abstract — **In this paper, the synthesis problem of reduced-order H_2 dynamic output feedback controllers with coefficient constraint is considered. Sufficient LMI conditions for the existence of such controllers are given and explicit formulas for the controllers are derived. Numerical examples are provided to demonstrate the effectiveness of the proposed method.**

Index Terms — **Robust control, H_2 control, reduced-order, controller coefficient constraint, linear matrix inequality(LMI)**

I. Introduction

In the past two decades H_2 and H_∞ control have been active areas of research. While the synthesis methods for obtaining full-order controllers for both of the problems have reached a certain level of maturity, there still lacks an efficient algorithm for the reduced-order case even if the solutions do exist.

In the literature most of the papers relevant to the reduced-order case are concerned with H_∞ performance. The existence of reduced-order suboptimal H_∞ controllers can be fully characterized by a system of linear matrix inequalities together with a rank constraint [1], [2]. However, the mathematical problem is very difficult to solve. A number of optimization techniques have been proposed to attack this problem, see e.g., [1,3]. In [4] the problem was reformulated and solved by an alternative projection technique. In [5] the synthesis problem was converted into a static output feedback design problem. Sufficient LMI conditions were derived. Several other LMI-based methods were proposed [6-10] via employing different formulations and concepts. Despite the hot studies on the reduced-order H_∞ controller issue, little attention has been paid toward the corresponding H_2 problem [11].

On the other hand, it is known that the existing H_2 and H_∞ synthesis methods have no control on the magnitude of the resulting controllers' coefficients. Sometimes, the controllers obtained by these methods exhibit very large coefficients which are too big to be realized. Therefore, the purpose of this paper is to propose a design method for obtaining H_2 reduced-order controllers with coefficient constraint taken into consideration . We will present a LMI approach to tackle this problem. Due to space limitation, some of the proofs have been omitted.

II. Problem Formulation

We consider a linear, time-invariant, single-input-single-output (SISO) continuous-time control system as shown in Fig. 1.,

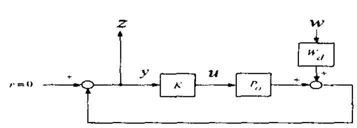

Fig. 1. Feedback control system with disturbance.

where P_o represents the nominal plant, W_d denotes a weighting function reflecting the frequency content of the disturbance w , and K is the dynamic output feedback controller to be designed. Our goal is to design a reduced-order, if possible, dynamic output feedback controller $K(s)$ with reasonable coefficients to achieve closed-loop stability as well as to attenuate the effect of the disturbance on the system as possible. This problem can be recast as a reduced-order H_2 control problem. To this end, it is routine to convert the conventional control system framework in Fig. 1. into the popular robust control framework as shown in Fig. 2.

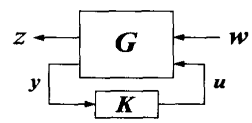

Fig. 2. $G-K$ framework.

For further exposition, we make the following assumption: both the nominal plant and the weighting function are strictly proper. Furthermore, without loss of generality we may assume that the generalized plant G is of dimension *n*. It follows that

$$Y(s) = \frac{\left(b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots b_1 s + b_0\right)}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} U(s) + \frac{\left(c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots c_1 s + c_0\right)}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0} W(s)$$

where $Y(s), U(s)$, and $W(s)$ are the Laplace transform of the signals y , u , and w , respectively. Multiplying every term by the common denominator and dividing every term by $(s+d)^{n-1}$, where d is a positive number, yield

$$\begin{aligned} sY(s) = \bar{a}_0 Y(s) + \frac{\bar{a}_1}{s+d}Y(s) + \cdots + \frac{\bar{a}_{n-1}}{(s+d)^{n-1}}Y(s) \\ + \bar{b}_0 U(s) + \frac{\bar{b}_1}{s+d}U(s) + \cdots + \frac{\bar{b}_{n-1}}{(s+d)^{n-1}}U(s) \\ + H(s)W(s). \end{aligned}$$

where

$$H(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots c_1 s + c_0}{(s+d)^{n-1}}.$$

We define a new state vector $\xi \in R^{2n-1}$ as follows.

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$$\xi(s)\equiv\left(Y(s),\frac{1}{s+d}Y(s),\frac{1}{s+d}U(s),\ \cdots\ \frac{1}{(s+d)^{n-1}}Y(s),\frac{1}{(s+d)^{n-1}}U(s)\right)^T$$

$$=[\xi_1(s),\ \xi_2(s),\ \ \xi_3(s),\ \ \cdots\ \ \xi_{2n-2}(s),\ \ \xi_{2n-1}(s)]^T.$$

which slightly generalizes the definition of the new state vector in [12]. The introduction of the new states decomposes the system T_{zw} in Fig. 2 into two subsystems $T_{z\bar{w}}$ and $H(s)$, in serial connection as shown in Fig. 3.

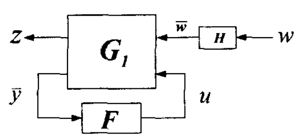

Fig. 3. State-feedback interconnected system.

where

$$G_1\begin{cases}\dot{\xi}=A_g\xi+B_{g1}\overline{w}+B_{g2}u\\ z=C_g\xi\\ \overline{y}=\xi\end{cases}$$

with

$$A_g=\begin{bmatrix}\bar{a}_0 & \bar{a}_1 & \bar{b}_1 & \bar{a}_2 & \bar{b}_2 & \cdots & \bar{a}_{n-2} & \bar{b}_{n-2} & \bar{a}_{n-1} & \bar{b}_{n-1}\\ 1 & -d & 0 & 0 & 0 & \cdots & \cdot & \cdot & \cdot & 0\\ 0 & 0 & -d & 0 & 0 & \cdots & \cdot & \cdot & \cdot & 0\\ 0 & 1 & 0 & -d & 0 & \cdots & \cdot & \cdot & \cdot & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0\\ 0 & 0 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdot & \cdot & \cdot & \cdots & 1 & 0 & -d & 0\\ 0 & 0 & \cdot & \cdot & \cdot & \cdots & 0 & 1 & 0 & -d\end{bmatrix}$$

$$B_{g1}=[1\quad 0\quad 0\quad 0\quad \cdots\ \cdots\quad 0\quad 0\quad 0\quad 0]^T$$
$$B_{g2}=[\bar{b}_0\quad 0\quad 1\quad 0\quad \cdots\ \cdots\quad 0\quad 0\quad 0\quad 0]^T$$
$$C_g=[1\quad 0\quad 0\quad 0\quad \cdots\ \cdots\quad 0\quad 0\quad 0\quad 0]$$

Moreover, the static feedback control law $u=F\,\overline{y}=F\xi$ in Fig. 3 is equivalent to a dynamic output feedback controller of order no greater than $n-1$ in Fig. 2. Thus the original reduced-order H_2 dynamic controller synthesis problem turns out to be a state feedback gain design problem. Next, merging the $(n-1)th$ order filter $H(s)$ into $G_1(s)$ results in a static output feedback system as shown in Fig. 4.

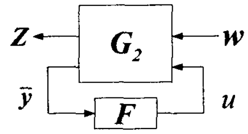

Fig. 4. Static output feedback scheme.

where

$$G_2\begin{cases}\dot{x}=Ax+B_1w+B_2u\\ z=C_1x\\ \overline{y}=C_2x\end{cases}\qquad(1)$$

where $x=[\xi',x_f']'$ is the augmented state vector consisting of both ξ and the filter state x_f. Thus the order of G_2 is $3n-2$ and $C_2=\left(I_{2n-1}\ \ 0_{(2n-1)\times(n-1)}\right)$. After closing the loop, i.e., consider $u=F\overline{y}=FC_2x$, the closed-loop system becomes

$$\begin{cases}\dot{x}=\left(A+B_2FC_2\right)x+B_1w\\ z=C_1x\end{cases}$$

The problem now becomes finding static output feedback gain $F=\left[f_1,\cdots,f_{2n-1}\right]$ with reasonable magnitude constraint, such that the closed-loop transfer function T_{zw} is stable and the performance $\|T_{zw}\|_2$ is minimized.

III. Main Results

In this section, we present a LMI approach to solve the reduced-order H_2 output feedback controller design problem mentioned above with and without considering magnitude constraint on the controller coefficients, respectively.

A. Reduced-order H_2 controller design without coefficient constraint

Lemma 1: Consider the system described by (1). Given $\nu\succ 0$. If there exist matrices $N\in R^{1\times(2n-1)}$, $M=M^T\in R^{(2n-1)\times(2n-1)}$, $W_{22}=W_{22}^T\in R^{(n-1)\times(n-1)}$ and $Q=Q^T\in R$ satisfying LMIs (2)-(4).

$$\begin{pmatrix}A\begin{pmatrix}M & 0\\ 0 & W_{22}\end{pmatrix}+B_2NC_2+\begin{pmatrix}M & 0\\ 0 & W_{22}\end{pmatrix}A^T+C_2^TN^TB_2^T & B_1\\ * & -I\end{pmatrix}\prec 0\qquad(2)$$

$$\begin{pmatrix}\begin{pmatrix}M & 0\\ 0 & W_{22}\end{pmatrix} & \left(C_1\begin{pmatrix}M & 0\\ 0 & W_{22}\end{pmatrix}\right)^T\\ * & Q\end{pmatrix}\succ 0\qquad(3)$$

$$\mathrm{Trace}(Q)\prec\nu^2.\qquad(4)$$

Then there exists a dynamic output feedback controller $K(s)$ of order no greater than $n-1$ which renders the closed-loop system in Fig. 1 stable and $\|T_{zw}\|_2\prec\nu$. Furthermore,

$$K(s)=\frac{f_1(s+d)^{n-1}+\sum_{j=1}^{n-1}f_{2j}(s+d)^{n-1-j}}{(s+d)^{n-1}-\sum_{j=1}^{n-1}f_{2j+1}(s+d)^{n-1-j}}$$

where $\left[f_1,\cdots,f_{2n-1}\right]=NM^{-1}$.

Proof: Given $\nu \succ 0$. The closed-loop system T_{zw} in Fig.1. is stable with $\| T_{zw} \|_2 \prec \nu$ if and only if there exist matrices $P = P^T$, $Q = Q^T$ and output feedback gain F with appropriate dimensions satisfying the following BMIs.

$$\begin{pmatrix} P(A + B_2FC_2) + (A + B_2FC_2)^T P & PB_1 \\ * & -I \end{pmatrix} \prec 0$$

$$\begin{pmatrix} P & C_1^T \\ * & Q \end{pmatrix} \succ 0$$

$$\text{Trace } (Q) \prec \nu^2.$$

Let $W = P^{-1}$ and perform congruence transform $diag(W, I)$ on the first two BMIs. The above condition turns out to be the existence of matrices $W = W^T$, $Q = Q^T$ and F such that the following BMIs hold.

$$\begin{pmatrix} (A + B_2FC_2)W + W(A + B_2FC_2)^T & B_1 \\ * & -I \end{pmatrix} \prec 0$$

$$\begin{pmatrix} W & (C_1W)^T \\ * & Q \end{pmatrix} \succ 0$$

$$\text{Trace } (Q) \prec \nu^2.$$

Assume $F = NM^{-1}$ where N and M are matrix variables. Under the proposed condition $C_2W = MC_2$ [13] which is equivalent to $W = \begin{pmatrix} M & 0 \\ 0 & W_{22} \end{pmatrix}$ because $C_2 = \begin{pmatrix} I_{2n-1} & 0_{(2n-1)\times(n-1)} \end{pmatrix}$, it follows that $B_2FC_2W = B_2NC_2$. Substituting these into the above BMIs; it is easy to check that the assertion is proved. The formula for the resultant controller is easily derived via the definition of the state vector. □

In case f_{2n-1} and f_{2n-2} are not both zero, then the controller is of order $n-1$. Extension to obtain controllers of order less than $n-1$ is straightforward via controlling the number of the variables of the feedback gain vector F.

Proposition 1: Under the same premise as that of Lemma 1. For each integer $i = 0, 1, 2, \cdots, n-1$ if there exist matrices $N \in R^{1\times(2n-1-2i)}$, $M = M^T \in R^{(2n-1-2i)\times(2n-1-2i)}$, $W_{22} = W_{22}^T \in R^{(n-1+2i)\times(n-1+2i)}$, and $Q = Q^T \in R$ satisfying LMIs (2)-(4) with $C_2 = \begin{pmatrix} I_{2n-1-2i} & 0_{(2n-1-2i)\times(n-1+2i)} \end{pmatrix}$. Then there exists a dynamic output feedback controller $K(s)$ of order no greater than $n-i-1$, which renders the closed-loop system in Fig. 1. stable and $\| T_{zw} \|_2 \prec \nu$. Furthermore,

$$K(s) = \frac{f_1(s+d)^{n-1-i} + \sum_{j=1}^{n-1-i} f_{2j}(s+d)^{n-1-i-j}}{(s+d)^{n-1-i} - \sum_{j=1}^{n-1-i} f_{2j+1}(s+d)^{n-1-i-j}} \tag{5}$$

where $[f_1, \cdots, f_{2n-1-2i}] = NM^{-1}$.

The optimal H_2 performance achievable by this approach can be obtained by performing bisection method on ν or directly using LMI toolbox command "mincx".

B. Reduced-order H_2 controller design with coefficient constraint

The following lemma is applicable to convert magnitude constraint on the controller coefficients into LMI conditions.

Lemma 2: Given $\beta \succ 0$. Let F, N, and M be matrices with $F = NM^{-1}$ and $M \succ 0$. Then $\| F \| \prec \beta$ if there exist a scalar α satisfying the following LMIs.

$$\begin{pmatrix} \alpha I & N \\ N^T & \alpha I \end{pmatrix} \succ 0 \tag{6}$$

$$\alpha I \prec \beta M. \tag{7}$$

With the lemma at hand, we are in the position to state another result which give a sufficient condition for the existence of reduced-order H_2 controller with a prescribed magnitude constraint on the controller coefficients.

Theorem 1: Consider the system described by (1). Given performance level $\nu \succ 0$ and magnitude constraint $\beta \succ 0$. For each integer $i = 0, 1, 2, \cdots, n-1$ if there exist matrices $N \in R^{1\times(2n-1-2i)}$, $M = M^T \in R^{(2n-1-2i)\times(2n-1-2i)}$, $W_{22} = W_{22}^T \in R^{(n-1+2i)\times(n-1+2i)}$ and scalar α satisfying the LMIs (2)-(4) and (6)-(7). Then there exists a dynamic output feedback controller $K(s)$ of order no greater than $n-i-1$ described by (5) with coefficient constraint $\| [f_1, \cdots, f_{2n-1-2i}] \| \prec \beta$, which renders the closed-loop system in Fig. 1. stable and $\| T_{zw} \|_2 \prec \nu$.

IV. NUMERICAL EXAMPLES

Example 1: Consider the system as depicted in Fig. 1, where the transfer function $P_0(s)$ is taken from [14]:

$$P_0(s) = \frac{54s + 90}{s^4 + 2.8s^3 + 50s^2 + 30s - 0.1}; \quad W_d = \frac{1}{s+1}$$

Using the LMI control toolbox [15] yields the optimal H_2 performance 0.0182, and the corresponding controller is given by

$$K_{opt}(s) = \frac{10^5 \times (4.3s^4 - 6.25\times10^6 s^3 + 5.52\times10^7 s^2 - 1\times\times10^8 s - 4.88\times10^6)}{s^5 + 2.6\times10^6 s^4 + 2.9\times10^7 s^3 + 5.1\times10^7 s^2 + 1.4\times10^8 s + 8.5\times10^7}$$

which is of order 5 and has extremely large controller coefficients. Applying our method(with ad hoc choice $d = 1$) without taking coefficient constraints into consideration, i.e., we solve the minimization problem : Minimize ν subject to LMIs (2)-(4), where ν represents the upper bound of $\| T_{zw} \|_2$. The results are shown in the 2nd column of Tab. 1.

Tab. 1. The H2 performance with and without coefficient constraint

Order of *K(s)*	ν^*	ν^{**}
4	0.0189	0.1551
3	0.0200	0.1692
2	0.0221	0.5822
1	0.6721	0.6025

where the values ν^* are the H2 performance of the closed-loop systems and the resultant controllers of the individual cases are given by

$$K_4(s)=\frac{10^{11}\times(5.39s^4+2144s^3+3214s^2+2148s+5.39)}{s^4+2.97\times10^6 s^3+1.32\times10^{10}s^2+3.51\times10^{10}s+2.19\times10^{10}}$$

$$K_3(s)=\frac{10^{11}\times(5.36s^3+1596s^2+16s+5.36)}{s^3+2.89\times10^6 s^2+1.29\times10^{10}s+2.15\times10^{10}}$$

$$K_2(s)=\frac{10^{11}\times(2.15s^2+4.27s+2.22)}{s^2+1.83\times10^6 s+8.24\times10^9}$$

$$K_1(s)=\frac{10^5\times(5655s+2.85)}{s+2.91\times10^5}$$

In order to make the controller coefficients more reasonable while approximately preserving the performance, we solve the following generalized eigenvalue minimization problem (GEVP): Set the value ν in (4) as close to ν^* as possible, minimize β subject to LMIs (2)-(4) and (6)-(7), where β represents a magnitude constraint on the controller coefficients. The reduced-order controllers obtained are as follows.

$$K_4(s)=\frac{10^4\times(7.76\,s^4+26.36\,s^3+39.01s^2+28.17s+7.77)}{s^4+240.4\,s^3+54120\,s^2+143000s+89140}$$

$$K_3(s)=\frac{10^4\times(4.80\,s^3+11.19s^2+12.43s+4.813)}{s^3+181.1s^2+27320s+61690}$$

$$K_2(s)=\frac{250.6\,s^2-86.04s+802.1}{s^2+48016s+1411}$$

$$K_1(s)=\frac{17020s+14690}{s+7326}$$

Note that, while the performance ν^{**} becomes worse a little, the controller coefficients are indeed much smaller than that in the unconstrained case.

Example 2: In this example, we propose to design a controller, preferably reduced-order, not only to enlarge stability margin of the system against perturbation Δ, but also to attenuate the effect of the disturbance on the system output. This problem can be cast as a mixed H_2/H_∞ problem as shown in Fig. 5. Assume that the plant and the

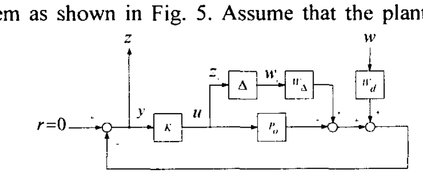

Fig. 5. Perturbed system with external disturbance

weighting functions are the same as example1. We solve the linear objective minimization problem: Minimize $\|T_{z_1w_1}\|_\infty+\|T_{z_2w_2}\|_2^2$ by LMI toolbox and our method in which magnitude constraint is not considered. In both cases situation of large controller coefficients remains. Again, by our method which takes the magnitude constraint into consideration (set magnitude constraint $\beta=100$) we obtain controllers of order from 5 down to 1 with comparable performances and much smaller controller coefficients.

V. CONCLUSIONS

In this paper we dealt with the reduced-order H_2 dynamic output feedback controller synthesis problem which has largely been ignored for a long time. Sufficient conditions for the existence of the controllers were derived. In addition, in response to the unusually large coefficients of the resulting H_2 controllers in many cases, magnitude constraint on the controller coefficients was taken into consideration during the design process. These conditions are all in LMI form which can be efficiently solved via existing software. Numerical results show that the controllers found by our method indeed exhibit much lower coefficients.

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